

# Math 210A Lecture 4 Notes

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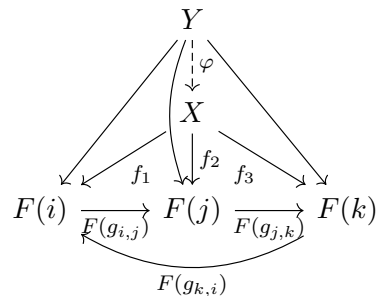
October 5, 2018

## 1 Limits and colimits

### 1.1 Limits

**Definition 1.1.** Let  $I$  be a small index category, and let  $F : I \rightarrow \mathcal{C}$  be a functor. A **limit**  $\lim F$  is an object  $X$  with morphisms  $f_i : X \rightarrow F(i)$ , characterized by the following properties:

1. If  $g_{i,j} : F(i) \rightarrow F(j)$  is a morphism, then  $f_j = F(g_{i,j}) \circ f_i$ .
2. Any  $Y$  with this property factors through  $X$ ; i.e. there exists a unique  $\varphi : Y \rightarrow X$  such that  $f'_i = f_i \circ \varphi$  for all  $i$ .



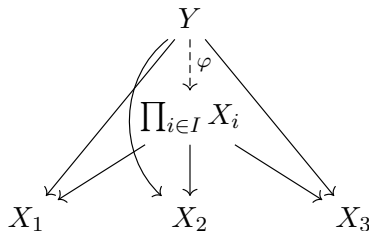
The second property is called a **universal property**.

**Remark 1.1.** The limit includes the data of the  $f_\alpha$  maps.

**Proposition 1.1.** *If it exists,  $\lim F$  is unique up to isomorphism. Moreover, this isomorphism is unique,*

*Proof.* Suppose  $(X, \{f_\alpha\})$  and  $(Y, \{f'_\alpha\})$  are both limits of  $F$ . Since both of them are limits, let  $\phi : X \rightarrow Y$  and  $\psi : Y \rightarrow X$  be the unique maps given by the universal property.  $\square$

**Definition 1.2.** Let  $I$  be a discrete category (only identity morphisms). Then  $F : I \rightarrow \mathcal{C}$  is determined by a collection  $(X_i)_{i \in I}$  of objects. Then the **product** is  $\prod_{i \in I} X_i = \lim F$ .



For the morphisms, we have

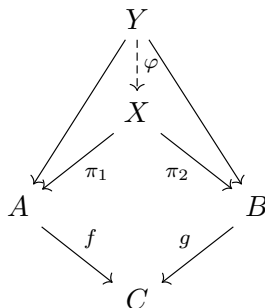
$$\mathrm{Hom}_{\mathcal{C}}(Z, \prod X_i) \simeq \prod \mathrm{Hom}_{\mathcal{C}}(Z, X_i).$$

**Example 1.1.** In the category of sets, the product is the set-theoretic product.

**Example 1.2.** In  $\mathrm{Ab}$ ,  $\mathrm{Grp}$ , and  $\mathrm{Mod}$ , the product is the usual product, as well.

**Example 1.3.** In  $\mathcal{C} = \mathrm{Fld}$ , the product is not the usual product.  $\mathbb{Q} \times \mathbb{Q}$  is not a field. You can also check that the product of  $\mathbb{Q}(\sqrt{2})$  and  $\mathbb{Q}(\sqrt{2})$  does not exist.

**Definition 1.3.** The *pull-back*  $X = A \times_C B$  is a limit of  $A$  and  $B$  with morphisms  $f : A \rightarrow C$  and  $g : B \rightarrow C$ .



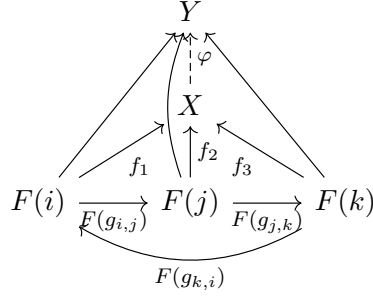
**Remark 1.2.** Even though we write the pull-back as  $X = A \times_C B$ , it depends on the morphisms  $f, g$ .

**Example 1.4.** In  $\mathrm{Set}$ , the pullback is  $A \times_C B = \{(a, b) \in A \times B : f(a) = g(b)\}$ .

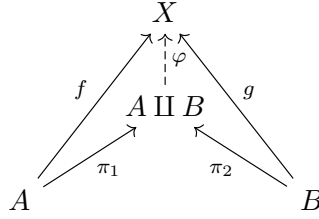
## 1.2 Colimits

**Definition 1.4.** Let  $I$  be a small index category, and let  $F : I \rightarrow \mathcal{C}$  be a functor. A **colimit**  $\mathrm{colim} F$  is an object  $X$  with morphisms  $f_i : F(i) \rightarrow X$ , characterized by the following properties:

1. If  $g_{i,j} : F(i) \rightarrow F(j)$  is a morphism, then  $f_i = f_j \circ F(g_{i,j})$ .
2. Any  $Y$  with this property factors through  $X$ ; i.e. there exists an unique  $\varphi : Y \rightarrow X$  such that  $f'_i = \varphi \circ f_i$  for all  $i$ .



**Definition 1.5.** Let  $I$  be a discrete category (only identity morphisms). Then  $F : I \rightarrow \mathcal{C}$  is determined by  $(A_i)_{i \in I}$ . Then the **coproduct** is  $\coprod_{i \in I} X_i = \text{colim } F$ .



**Example 1.5.** In the category of sets, the coproduct is the disjoint union.

**Example 1.6.** In the category of groups,  $G_1 \amalg G_2$  is call the **free product** of  $G_1, G_2$ . This is usually denoted by  $G_1 * G_2$ .

**Example 1.7.** In the category of  $R$ -modules,

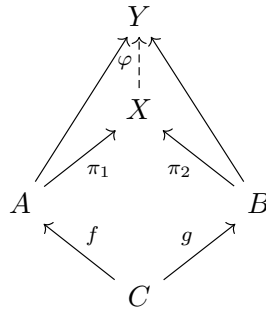
$$\prod_{i \in I} M_i = \bigoplus_{i \in I} M_i = \left\{ \sum_{i=1}^n r_i m_i : r_i \in R, m_i \in M_i \right\}.$$

If  $I$  is infinite, this is not the same as

$$\prod_{i \in I} M_i = \{(r_i m_i)_{i \in I} : r_i \in R, m_i \in M_i\}.$$

**Example 1.8.** In the category of commutative rings,  $R \amalg A = R \otimes_{\mathbb{Z}} S$ .

**Definition 1.6.** The *push-out*  $X = A \amalg_C B$  is a colimit of  $A$  and  $B$  with morphisms  $f : C \rightarrow A$  and  $g : C \rightarrow B$ .



**Example 1.9.** In  $\text{Set}$ ,  $Y \amalg_C Z = \{x \in Y \amalg Z : f(x) = g(x)\}$ .

**Example 1.10.** In the category of groups,  $G_1 \amalg_H G_2$  is called the amalgamated free product and is denoted by  $G_1 *_H G_2$ .

**Example 1.11.** In the category of commutative rings,  $S_1 \amalg_R S_2 = S_1 \otimes_R S_2$ .

**Definition 1.7.** If  $\lim F$  exists, we say  $\mathcal{C}$  **admits the limit** of  $F$ . If  $\mathcal{C}$  admits all (small) limits, we say  $\mathcal{C}$  is **complete**. If  $\mathcal{C}$  admits all (small) colimits,  $\mathcal{C}$  is **cocomplete**.

**Example 1.12.** The category of sets is both complete and cocomplete.