Math 210A Lecture 4 Notes

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October 5, 2018

1 Limits and colimits

1.1 Limits

Definition 1.1. Let I be a small index category, and let $F : I \to \mathcal{C}$ be a functor. A **limit** $\lim F$ is an object X with morphisms $f_i : X \to F(i)$, characterized by the following properties:

1. If $g_{i,j} : F(i) \to F(j)$ is a morphism, then $f_j = F(g_{i,j}) \circ f_i$.

2. Any Y with this property factors through X; i.e. there exists an unique $\varphi : Y \to X$ such that $f_i' = f_i \circ \varphi$ for all i.

$$\begin{tikzcd}
 & Y \arrow[d, dashed, "\varphi"] \arrow[ddl] \arrow[dd, bend right] \arrow[ddr] & \\
 & X \arrow[dl, "f_1"'] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\
F(i) \arrow[r, "F(g_{i,j})"'] & F(j) \arrow[r, "F(g_{j,k})"'] & F(k) \arrow[ll, bend left, "F(g_{k,i})"]
\end{tikzcd}$$

The second property is called a **universal property**.

Remark 1.1. The limit includes the data of the f_α maps.

Proposition 1.1. *If it exists,* $\lim F$ *is unique up to isomorphism. Moreover, this isomorphism is unique,*

Proof. Suppose $(X, \{f_\alpha\})$ and $(Y, \{f_\alpha'\})$ are both limits of F. Since both of them are limits, let $\phi : X \to Y$ and $\psi : Y \to X$ be the unique maps given by the universal property. $\square$

Definition 1.2. Let I be a discrete category (only identity morphisms). Then $F : I \to \mathcal{C}$ is determined by a collection $(X_i)_{i \in I}$ of objects. Then the **product** is $\prod_{i \in I} X_i = \lim F$.

$$\begin{tikzcd}
 & Y \arrow[d, dashed, "\varphi"] \arrow[ddl] \arrow[ddr] \arrow[dd, bend right] & \\
 & \prod_{i \in I} X_i \arrow[dl] \arrow[d] \arrow[dr] & \\
X_1 & X_2 & X_3
\end{tikzcd}$$

For the morphisms, we have

$$\mathrm{Hom}_{\mathcal{C}}(Z, \prod X_i) \simeq \prod \mathrm{Hom}_{\mathcal{C}}(Z, X_i).$$

Example 1.1. In the category of sets, the product is the set-theoretic product.

Example 1.2. In Ab, Grp, and Mod, the product is the usual product, as well.

Example 1.3. In $\mathcal{C} = \mathrm{Fld}$, the product is not the usual product. $\mathbb{Q} \times \mathbb{Q}$ is not a field. You can also check that the product of $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})$ does not exist.

Definition 1.3. The *pull-back* $X = A \times_C B$ is a limit of A and B with morphisms $f : A \to C$ and $g : B \to C$.

$$\begin{tikzcd}
 & Y \arrow[d, dashed, "\varphi"] \arrow[ddl] \arrow[ddr] & \\
 & X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & \\
A \arrow[dr, "f"] & & B \arrow[dl, "g"] \\
 & C &
\end{tikzcd}$$

Remark 1.2. Even though we write the pull-back as $X = A \times_C B$, it depends on the morphisms f, g.

Example 1.4. In Set, the pullback is $A \times_C B = \{(a, b) \in A \times B : f(a) = g(b)\}$.

1.2 Colimits

Definition 1.4. Let I be a small index category, and let $F : I \to \mathcal{C}$ be a functor. A **colimit** $\mathrm{colim}\, F$ is an object X with morphisms $f_i : F(i) \to X$, characterized by the following properties:

1. If $g_{i,j} : F(i) \to F(j)$ is a morphism, then $f_i = f_j \circ F(g_{i,j})$.
2. Any Y with this property factors through X; i.e. there exists an unique $\varphi : Y \to X$ such that $f_i' = \varphi \circ f_i$ for all i.

$$\begin{tikzcd}
 & Y & \\
 & X \arrow[u, dashed, "\varphi"'] & \\
F(i) \arrow[uur] \arrow[ur, "f_1"'] \arrow[r, "F(g_{i,j})"'] & F(j) \arrow[u, "f_2"'] \arrow[uu, bend left] \arrow[r, "F(g_{j,k})"'] & F(k) \arrow[uul] \arrow[ul, "f_3"] \arrow[ll, bend left, "F(g_{k,i})"]
\end{tikzcd}$$

Definition 1.5. Let I be a discrete category (only identity morphisms). Then $F : I \to \mathcal{C}$ is determined by $(A_i)_{i\in I}$. Then the **coproduct** is $\amalg_{i\in I} X_i = \operatorname{colim} F$.

$$\begin{tikzcd}
 & X & \\
 & A \amalg B \arrow[u, dashed, "\varphi"'] & \\
A \arrow[uur, "f"] \arrow[ur, "\pi_1"'] & & B \arrow[uul, "g"'] \arrow[ul, "\pi_2"]
\end{tikzcd}$$

Example 1.5. In the category of sets, the coproduct is the disjoint union.

Example 1.6. In the category of groups, $G_1 \amalg G_2$ is call the **free product** of G_1, G_2. This is usually denoted by $G_1 * G_2$.

Example 1.7. In the category of R-modules,

$$\coprod_{i\in I} M_i = \bigoplus_{i\in I} M_i = \left\{ \sum_{i=1}^{n} r_i m_i : r_i \in R, m_i \in M_i \right\}.$$

If I is infinite, this is not the same as

$$\prod_{i\in I} M_i = \{(r_i m_i)_{i\in I} : r_i \in R, m_i \in M_i\}.$$

Example 1.8. In the category of commutative rings, $R \amalg A = R \otimes_{\mathbb{Z}} S$.

Definition 1.6. The *push-out* $X = A \amalg_C B$ is a colimit of A and B with morphisms $f : C \to A$ and $g : C \to B$.

$$\begin{tikzcd}
 & Y & \\
 & X \arrow[u, dashed, "\varphi"] & \\
A \arrow[ur, "\pi_1"] \arrow[uur] & & B \arrow[ul, "\pi_2"'] \arrow[uul] \\
 & C \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Example 1.9. In Set, $Y \amalg_C Z = \{x \in Y \amalg Z : f(x) = g(x)\}$.

Example 1.10. In the category of groups, $G_1 \amalg_H G_2$ is called the amalgamated free product and is denoted by $G_1 *_H G_2$.

Example 1.11. In the category of commutative rings, $S_1 \amalg_R S_2 = S_1 \otimes_R S_2$.

Definition 1.7. If $\lim F$ exists, we cay $\mathcal{C}$ **admits the limit** of F. If $\mathcal{C}$ admits all (small) limits, we cay $\mathcal{C}$ is **complete**. If $\mathcal{C}$ admits all (small) colimits, $\mathcal{C}$ is **cocomplete**.

Example 1.12. The category of sets is both complete and cocomplete.